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PREFACE.

There must be something not quite sound about the teaching of Trigonometry in England when students who freely use circular measure do not learn how to find the number of degrees in their unit. Nor is it altogether satisfactory that the higher students should have to find the ratio of the circumference of a circle to the diameter by reasoning involving impossible quantities. Such reasoning is correct, and to most mathematicians convincing; but there have been many, and among them some eminent thinkers, to whom it has been incomprehensible. And yet in our text-books is to be found no word of the only method known till Demoivre's time of approximating to the most important constant in Mathematics.

Some strong reason is required to justify the banishment of the historical interest* from Trigonometry. Is there such? The old method of evaluating π is not less interesting nor less instructive than the new. To the degree of accuracy required in practice it is not much more laborious than the other. It shews how Trigonometrical tables have been calculated. It introduces the idea of a limit. And few pieces of Geometrical reasoning are more elegant than Huyghens' proofs given in the 17th and 18th Articles of this "account" and those which lead up to them.

The subject as begun by Archimedes and finished by Huyghens is complete in itself.

I have put together the principal propositions in a modern form and shewn their connection with Trigonometry, and I venture to appeal to Teachers and Authors whether some of them ought not to find a place on the threshold of the subject.

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* Those who have read De Morgan's *Budget of Paradoxes* or the article "Quadrature of the Circle" in the *English Cyclopædia* will think that I might have added "the amusement."

1. THE area of a circle is evidently greater than that of any polygon inscribed in it, and is evidently less than that of any polygon circumscribed to it. But, supposing the polygons regular, their areas are nearer to that of the circle and to one another the more numerous their sides are.

(It is easy to shew by the application of Euc. I. 20 that the perimeter of an inscribed polygon increases, and that of a circumscribed polygon diminishes as the number of sides becomes greater.)

From such considerations it follows that a circle is to be regarded as in its properties intermediate between a regular inscribed and circumscribed polygon of many sides, and identical with either when the number of sides is very great, or each side very small compared with the radius.

Thus we speak of a railway curve although each rail may be straight.

2. Two problems occupied the attention of mathematicians from very early times, viz. to rectify

the circle, (i. e.) to find a straight line equal in length to the circumference of a circle whose diameter is known; and to square the circle, (i. e.) to find a square whose area is equal to that of a circle whose diameter is known. It is now known that no integer or terminating fraction will express the ratio of the circumference to the diameter. In other words, these problems are incapable of perfectly accurate solution.

This is no isolated case in mathematics, but "a priori" considerations on the nature of the circle and the fitness of things have led many, both ancients and moderns, to believe that the ratio should be expressible by a simple number.

These problems will now be seen to be the same. For by dividing the circle or polygon into a great number of triangles by lines drawn from the centre, it appears that the area of a circle is equal to that of a triangle whose base is equal in length to the circumference and whose altitude is equal to the radius.

*This is the first proposition in Archimedes' book on "the measure of the circle."

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* The most easily accessible translation of the works of Archimedes into a modern language is that of Peyrard, into French. There are many Latin translations.

3. We shall know the length of the circumference of one of these many-sided polygons if we can find one side. This is done by starting from polygons whose properties we know. Thus the side of an inscribed hexagon is equal to the radius. This is equivalent to saying that

$$\sin 30^0 = \frac{1}{2}.$$

The side of the circumscribed hexagon is equal to $\frac{2}{\sqrt{3}} \times$ radius. This is equivalent to

$$\tan 30^0 = \frac{1}{\sqrt{3}}.$$

Now we possess means of finding the values of functions of half an angle from those of the whole. We have such formulæ as

$$2\cos^2\frac{A}{2} = 1 + \cos A, \quad 2\sin^2\frac{A}{2} = 1 - \cos A,$$

$$\tan^2\frac{A}{2} = \frac{1 - \cos A}{1 + \cos A}.$$

Finding the sine of half the angle is equivalent to finding the length of the side of the inscribed polygon of twice as many sides. Finding the tangent of half the angle is equivalent to finding the side of the circumscribed polygon of twice as many sides. Thus we can find the sides of the inscribed and circumscribed dodecagons. Whence those of the 24-sided figures; whence those of the 48-sided figures, and so on. If now we have found $\sin\frac{A}{2}$ where $nA = 360$, and n is large, we reason thus: one side of the inscribed polygon is

$$2 \times \text{radius} \times \sin\frac{A}{2},$$

and the perimeter of the inscribed polygon is

$$2n \times \text{radius} \times \sin\frac{A}{2}.$$

So the perimeter of the circumscribed polygon of n sides is

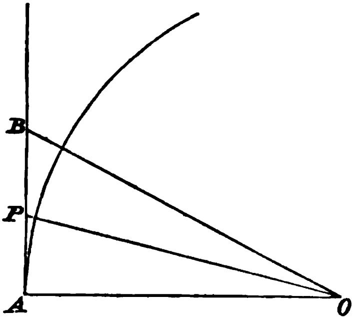
$$2n \times \text{radius} \times \tan\frac{A}{2}.$$

And since the circumference of the circle lies between these, we have two very near values between which the circumference must lie. Thus we get the ratio correct to so many decimal places.

4. The method of approximating to the side of a circumscribed polygon from knowing that of one of half the number of sides is thus given by Archimedes.

Let AOB be the third part of a right angle.

Then $OB : BA :: 306 : 153$ and the ratio $OA : AB$ is greater than $265 : 153$ (being $\sqrt{3} : 1$)*.



Bisect the angle AOB by the line OP.

Then shall $OB : OA :: BP : PA$ (Euclid VI. 3), or sum of OB and $OA : OA :: BA : AP$ ...(1), or sum of OB and $OA : BA :: OA : AP$.

Hence the ratio $OA : AP$ is greater than $571 : 153$.

Now AB is half the side of a circumscribed hexagon, and AP is half that of a circumscribed dodecagon.

(The trigonometrical formula to which (1) is equivalent is

$$\sec A + 1 : 1 :: \tan A : \tan \frac{A}{2}).$$

A somewhat similar method is pursued in the case of the inscribed figures.

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* These numbers are found by expressing $\sqrt{3}$ as a continued fraction. They may of course be verified by extracting the root.

5. From the above result Archimedes deduced, by a repetition of the process, that the ratio

radius : half the side of a circumscribed figure of 96 sides is greater than $4673\frac{1}{2} : 153$;

or the ratio,

diameter : side is greater than $4673\frac{1}{2} : 153$.

Hence the ratio of the whole perimeter of the polygon to the radius is less than $96 \times 153 : 4673\frac{1}{2}$,

$$\text{(i. e.) than } 3\frac{667\frac{1}{2}}{4673\frac{1}{2}} : 1,$$

$$\text{(i. e.) than } 3\tfrac{1}{7} : 1.$$

By means of the inscribed polygon of the same number of sides he shews that it is greater than $3\frac{10}{71} : 1$.

6. In trigonometrical language the above work has for its object to obtain the tangent (or sine) of a small angle. The side of a 96-sided polygon subtends at the centre an angle of $3\frac{3}{4}^{0}$. In finding the ratio of half the side to the radius Archimedes found the tangent and sine of $1^{o}{\cdot}875$.

Then he reasoned

$$\tan 1^{o}{\cdot}875 = \frac{\frac{1}{2}\text{ side of polygon}}{\text{radius}} = \frac{\frac{1}{2}\text{ arc}}{\text{radius}}, \text{ nearly.}$$

$$= \frac{\frac{1}{96}\text{ circumference}}{\text{diameter}},$$

whence the required ratio is known.

7. In those days there were no tables of sines or tangents. In fact, in order to construct these, it was necessary to find the sines and tangents of small angles by some such method as that which we have mentioned. If the student takes from such a table the sine or tangent of a small angle and multiplies it by the number of times the angle is contained in two right angles, he will obtain with very little work a much more accurate approximation than the above to the ratio—circumference : diameter.

[If he chooses to work without such a table he may still, with moderate calculation, obtain a result as near as that given above. The sine of the angle $1^\circ{\cdot}5$ is deducible from those of $37^\circ{\cdot}5$ and 36°. The former is the half of 75°. The sines and cosines and tangents of 75° and 36° are known from elementary Trigonometry.

$$\sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Now $$\sqrt{3} = 1{\cdot}7320,$$

$$\sqrt{2} = 1{\cdot}4142,$$

so that $$\sin 75^\circ = {\cdot}9659,$$

$$\sin 37^\circ{\cdot}5 = {\cdot}608.$$

Again, $\sin 36^\circ = \dfrac{\sqrt{10 - 2\sqrt{5}}}{4} = \cdot 587,$

whence $\sin 1^\circ\cdot 5 = \cdot 026,$

whence $\pi = \cdot 026 \times \dfrac{180}{1\cdot 5} = 3\cdot 14$ nearly.

If he wishes to work with tangents,

$$\tan 1^{0}\cdot 5 = \frac{\sin 1^{0}\cdot 5}{\sqrt{1 - \sin^2 1^{0}\cdot 5}}.$$

There are two sources of error here; the inaccuracy in the values of the sines and tangents and the difference of the arc from either. Both are avoided by carrying the operations to so many places that the results differ in the last figure.]

8. It is obvious that there is no limit to the accuracy attainable by this method except that imposed by the shortness of human life. Until the 16th century, mathematicians knew of no other method than this, and accordingly all approximations were obtained by additional halving of angles*. The most noticeable were the approximation of Adrianus

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* The history of the subject will be found in Montucla and in the *English Cyclopædia*, under the Article "Quadrature of the Circle." There are two interesting historical notes by Mr J. W. L. Glaisher in the *Messenger of Mathematics*, Vol. II. p. 127, and Vol. III. p. 27.

Romanus, who went to 14 places of decimals, and that of Ludolph van Ceulen, who went by 60 repetitions of the halving process to 35 places.

This involved immense labour, and van Ceulen was so proud of his work that he requested that these figures might be inscribed on his tombstone.

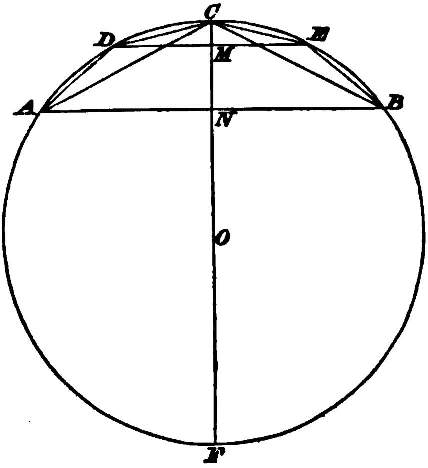
9. Thus far mathematicians had simply carried out the idea of Archimedes. A professor at Leyden called Willebrord Snell, who flourished in the beginning of the 17th century, was the first to search for closer approximations to an arc than the sides of the inscribed and circumscribed polygons.

He found two such lines which he was convinced were the one greater the other less than the arc. He was however unable to prove this. The necessary proof was supplied by Huyghens by means of the following train of propositions.

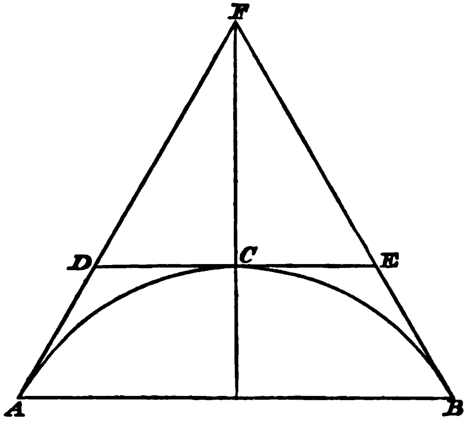
10. (1) *The maximum triangle inscribed in a segment of a circle* ACB *is less than four times the sum of the maximum triangles inscribed in the segments* AC *and* CB.

Let there be ADC and CEB.

The arcs AD, DC, CE, EB are evidently all equal. Join DE cutting CN in M. Then the triangle DCE is equal to either of the triangles DCA or BCE. Now the triangle DCE is equal to the rectangle $DM . MC$, while the triangle ABC is equal to the rectangle $AN . NC$. Hence the triangle ABC is to the triangle DCE as $AN . NC$ is to $DM . MC$. Now DM is equal to half AC (i. e.) it is greater than half AN; and MC is to NC as the square on CD is to the square on CA (for the square on DC is equal to the rectangle $CM . CF$) (Euclid VI. 8); and DC is greater than half AC; $\therefore$ MC is greater than one-fourth NC. Hence compounding these, the triangle ABC is less than eight times the triangle DCE, or four times the sum of ADC and CBE.



11. (2) *Two tangents and a third parallel to their chord of contact form a triangle which is greater than half the maximum triangle contained in the segment on the chord of contact.*



The triangle FED is greater than half the triangle ABC.

For DC is less than DF; ∴ AD is less than DF;

$$\therefore \ \Delta ABC < \frac{1}{2}\,\Delta ABF,$$

and $\Delta DEF : \Delta ABF$ is greater than 1 : 4 (Euclid VI. 19).

∴ ΔDEF is greater than half ΔABC.

12. From Prop. (1).

The circular area ACB is $= \Delta ACB +$ max$^{\text{m}}$. triangles in segments AC and CB + max^{m}. triangles in segments AD, DC, CE, EB + &c. to ∞,

$$> \Delta ACB + \frac{1}{4}\Delta ACB + \frac{1}{4}\cdot\frac{1}{4}\Delta ACB + \ldots.$$

$$> \frac{4}{3}\Delta ACB.$$

Whence (α) *the area of a circle is greater than the area of an inscribed polygon together with one-third of the excess of that polygon over one of half the number of sides.*

From Prop. (2).

The area between FA, FB and the curve is $> \frac{1}{2}$ the circular segment ACB,

whence the circular segment ACB is $< \frac{2}{3}$ the $\Delta\, FAB$,

whence (β) *the area of a circle is less than two-thirds of the area of a circumscribed polygon together with one-third of the corresponding inscribed polygon.*

13. In the figure Art. 10, join AO and DO. Then since the quadrilateral $AOCD$ is evidently equal to the triangle whose base is AC and whose altitude is the radius OD, the area of any inscribed polygon is equal to the triangle whose base is the perimeter of the polygon of half the number of sides and whose altitude is the radius. Since the area of the circle is equal to that of a triangle whose base is the circum-

ference, and whose altitude is the radius, it follows from (α) that

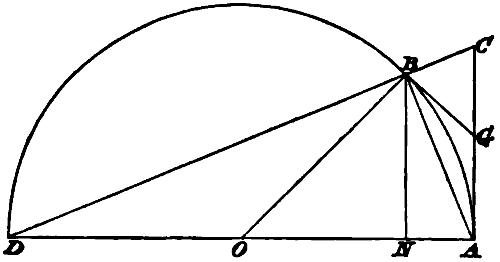
(α') *the circumference of any circle is greater than the perimeter of an inscribed polygon, together with one-third of the excess of the perimeter of that polygon over that of one of half the number of sides.*

Also (β') *if a straight line* DB *drawn through one end of a diameter intersect the tangent at the other end of the diameter in* C, *the arc* AB *will be less than*

$$\frac{2}{3}\text{CA} + \frac{1}{3}\text{BN}.$$

For draw BG touching the circle. Then

$$BG = CG = GA \text{ and } CA = BG + GA.$$



But (12, β) circular sector OAB

$$< \frac{2}{3} \text{ quadrilateral } OABG$$

$$+ \frac{1}{3} \text{ triangle } OAB,$$

$$\text{or arc } AB \times \text{radius} \quad < \frac{2}{3}(BG + GA)\,\text{radius},$$

$$+\frac{1}{3}BN\,.\,OA,$$

whence the proposition.

14. The trigonometrical formulæ corresponding to these are,

Area of circle or πr^2

$$\geq \frac{nr^2}{2}\sin\frac{A}{2} + \frac{1}{3}\left(\frac{nr^2}{2}\sin\frac{A}{2} - \frac{n}{2}\frac{r^2}{2}\sin A\right),$$

$$< \frac{2nr^2}{3}\tan\frac{A}{2} + \frac{1}{3}\frac{nr^2}{2}\sin A,$$

in which n is the number of sides and $A = \frac{360}{n} =$ number of degrees in the angle which a side of the polygon subtends at the centre.

15. One other proposition is needful before we can prove Snell's statements.

(γ) *The arc* AB *is less than*

$$\frac{2}{3}\text{BN} + \frac{1}{3}\text{MA}.$$

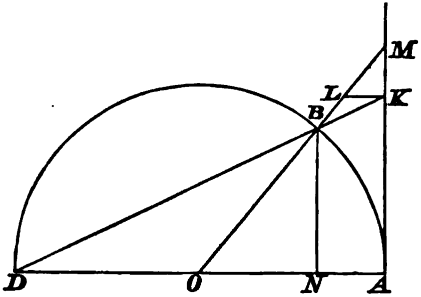
It has already been proved less than

$$\frac{2}{3}AK+\frac{1}{3}BN.$$

We will shew that

$$\frac{2}{3}AK+\frac{1}{3}BN<\frac{2}{3}BN+\frac{1}{3}AM,$$

(i.e.) $$2AK<BN+AM\ldots\ldots\ldots\ldots(1).$$



Draw KL parallel to the diameter. Then by similar triangles $BL=LK$.

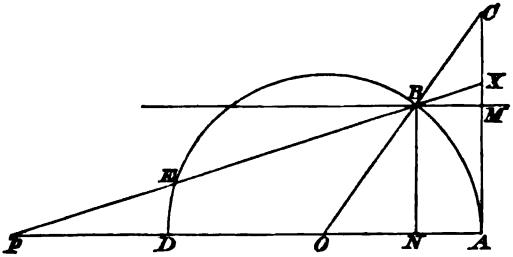
But $$LK<LM, \quad \therefore\ BL<LM,$$

$$\therefore\ 2OL<OB+OM.$$

Whence (1) follows.

16. The limits to the arc AB which the older mathematicians used were AC and BN. The only way of making these approach the arc is by diminishing the angle BOA, (i.e.) by increasing the

number of sides of the polygons of which AC and BN form parts. Snell looked on these as being both intercepts on the tangent,—the intercept AC by the radius produced and the intercept AM by a line through B from a very distant point on the diameter. Since then AM is less than the arc and AC greater, there must be some point in the diameter such as P whose intercept AX is equal to the arc. Snell found that this P is such that PD is nearly equal to the radius; and he convinced himself of the truth of the two following propositions:

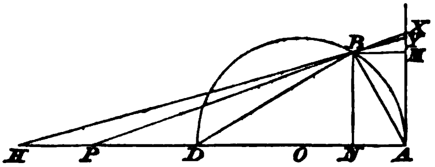


(1) *If* PD *is equal to the radius,* AX *is less than the arc* AB.

(2) *If* PE *is equal to the radius,* AX *is greater than the arc* AB.

The former of these is the more important. Anyone who reads Snell's attempts at proving these will allow that his results are probably true. But his proofs are manifestly based on a petitio principii. Both propositions have been proved by Huyghens in a treatise "de circulo magnitudine," a model of clear and elegant geometrical reasoning.

17. The following are his proofs:



(1) Take DH equal to DB. Join HB and produce it to meet the tangent at A in Y. Join BA. Then $\angle ABY = \angle H + \angle BAD$,

$$= \frac{1}{2}\angle BDA + \angle BAD.$$

$$= \text{complement of } \frac{1}{2} BDA,$$

$$= \angle AYB,$$

therefore AB is equal to AY, and YM is the excess of AB over BN.

If then we can shew that XY is less than one-third of YM we shall (Art. 12, a') have shewn that AX is less than the arc AB.

Since DN, DB and DA are in continued proportion, DB (i.e.) DH is < half of the sum of DA and DN. But DP is half DA.

$$\therefore \qquad HP < \text{half } DN.$$

Also $\qquad HP$ is $< PD$,

$\therefore$ three times $HP < DN + DP < NP$,

or $\qquad HP <$ one-third of PN.

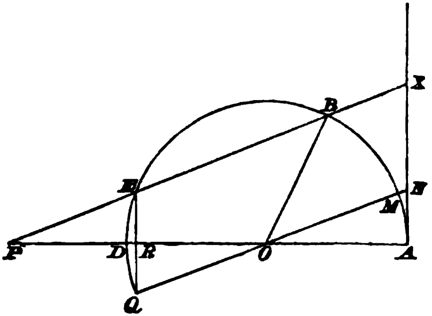
But by similar triangles

$$\text{and} \qquad \left.\begin{array}{l} XM : MB :: BN : NP \\ YM : MB :: BN : NH \end{array}\right\},$$

whence $\qquad XY : YM :: HP : PN$,

or $\qquad XY$ is < one-third YM.

18. Finally, to shew that when PE is equal to the radius, AX is > arc AB.



Draw $QOMN$ parallel to PE and join OE, EQ.

Then $\angle MOA$ is $\frac{1}{3} \angle BOA$.

$$\text{Now arc } AM < \frac{2}{3}RQ \text{ and } \frac{1}{3}AN \text{ (Art. 15)};$$

$$< \frac{1}{3}EQ \text{ and } \frac{1}{3}AN;$$

$$< \frac{1}{3}XN \text{ and } \frac{1}{3}AN;$$

$$< \frac{1}{3}XA;$$

whence arc $AB < XA$.

19. Let AOB be called A. Prop. (1) asserts that arc $AB > AX > \dfrac{BN.AP}{PN}$,

$$> \text{radius } \frac{3 \sin A}{2 + \cos A}.$$

Prop. (2) asserts that

$$\text{arc } AB < AX < AP \tan P. \quad \text{And } P = \frac{A}{3},$$

$$< (AO + OR + PR) \tan \frac{A}{3},$$

$$< \text{radius} \left(1 + 2 \cos \frac{A}{3}\right) \tan \frac{A}{3}.$$

To illustrate the great accuracy of these formulæ even with large angles, let us take $A = 30^0$, and use the former.

$$\text{arc of } 30^0 > \text{radius} \cdot \frac{\frac{3}{2}}{2 + \frac{\sqrt{3}}{2}},$$

whence $\pi > 3{\cdot}1404$.

If the angle $1^0{\cdot}5$ is used the value of π is correct to 8 decimal places. For this the sines must be known correct to 10 places.

No closer geometrical approximation has been obtained.

20. Those who are familiar with the expansions of $\sin\theta$ and $\cos\theta$ in powers of θ will recognize without difficulty that while the sides of polygons taken as approximations to the arc give

$$\sin\theta = \theta,$$

Snell's theorems give

$$\sin\theta = \theta - \frac{\theta^3}{6}.$$

Mr Glaisher shews that the numbers of decimal places to which the old method and Snell's method give the value of π correct when a polygon of n sides is used, are approximately the greatest integers in

$$2 \log_{10} n - 1{\cdot}190, \text{ and}$$

$$4 \log_{10} n - {\cdot}276,$$

respectively.

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